

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \dots \\ X_n \end{pmatrix} \quad - \quad n\text{-dimensional random vector} \quad \mathbf{E}(\mathbf{X}) = \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \dots \\ \mu_n \end{pmatrix}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{pmatrix} \quad \begin{aligned} \sigma_{ij} &= \text{Cov}(X_i, X_j) \\ \boldsymbol{\Sigma} &= \mathbf{E}[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})'] \\ \boldsymbol{\Sigma} &\text{ - symmetric, nonnegative-definite} \end{aligned}$$

Multivariate Normal Distribution:

$$\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\}$$

$$\mathbf{Z} \sim N_n(\mathbf{0}, \mathbf{I}_n) \quad f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}\mathbf{z}'\mathbf{z}\right\} = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z_i^2\right\}$$

$$\mathbf{X} = \boldsymbol{\Sigma}^{1/2} \mathbf{Z} + \boldsymbol{\mu}$$

$$M_{\mathbf{X}}(\mathbf{t}) = \exp\left\{\mathbf{t}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}\right\} \quad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ \dots \\ t_n \end{pmatrix} \in \mathbf{R}^n$$

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b} \quad \mathbf{A} - m \times n \quad \mathbf{b} \in \mathbf{R}^m$$

$$\Rightarrow \mathbf{Y} \sim N_m(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$$

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \quad \begin{aligned} \mathbf{X}_1 &\text{ is of dimension } m < n \\ \mathbf{X}_2 &\text{ is of dimension } n - m \end{aligned} \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

$$\mathbf{X}_1 | \mathbf{X}_2 \sim N_m\left(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} (\mathbf{X}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21}\right)$$

1. Consider a random vector $\vec{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ with mean $E(\vec{X}) = \vec{\mu} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$

and variance-covariance matrix $\text{Cov}(\vec{X}) = \Sigma = \begin{pmatrix} 9 & 2 & -3 \\ 2 & 4 & -2 \\ -3 & -2 & 16 \end{pmatrix}$.

Then $\text{Var}(X_1) = 9$, $\text{Var}(X_2) = 4$, $\text{Var}(X_3) = 16$,

$$\rho_{12} = \frac{2}{\sqrt{9} \cdot \sqrt{4}} = \frac{1}{3}, \quad \rho_{13} = \frac{-3}{\sqrt{9} \cdot \sqrt{16}} = -\frac{1}{4}, \quad \rho_{23} = \frac{-2}{\sqrt{4} \cdot \sqrt{16}} = -\frac{1}{4}.$$

Consider $2X_1 - 3X_2 - X_3 = \begin{pmatrix} 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$.

$$E(2X_1 - 3X_2 - X_3) = \begin{pmatrix} 2 & -3 & -1 \end{pmatrix} \vec{\mu} = 2\mu_1 - 3\mu_2 - \mu_3 = 2 \cdot 5 - 3 \cdot 1 - 1 \cdot 3 = 4.$$

$$\begin{aligned} \text{Var}(2X_1 - 3X_2 - X_3) &= \begin{pmatrix} 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 9 & 2 & -3 \\ 2 & 4 & -2 \\ -3 & -2 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 15 & -6 & -16 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 64. \end{aligned}$$

OR

$$\begin{aligned} \text{Var}(2X_1 - 3X_2 - X_3) &= 4 \text{Var}(X_1) + 9 \text{Var}(X_2) + \text{Var}(X_3) \\ &\quad - 12 \text{Cov}(X_1, X_2) - 4 \text{Cov}(X_1, X_3) + 6 \text{Cov}(X_2, X_3) \\ &= 36 + 36 + 16 - 24 + 12 - 12 = 64. \end{aligned}$$

2.

$$\mathbf{X} \sim N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad \boldsymbol{\mu} = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix}$$

a) Find $P(X_1 > 6)$.

$$X_1 \sim N(5, 4)$$

$$P(X_1 > 6) = P\left(Z > \frac{6-5}{\sqrt{4}}\right) = P(Z > 0.5) = \mathbf{0.3085}.$$

b) Find $P(5X_2 + 4X_3 > 70)$.

$$E(5X_2 + 4X_3) = 5 \cdot 3 + 4 \cdot 7 = 43.$$

$$\begin{aligned} \text{Var}(5X_2 + 4X_3) &= 25 \text{Var}(X_2) + 40 \text{Cov}(X_2, X_3) + 16 \text{Var}(X_3) \\ &= 25 \cdot 4 + 40 \cdot 2 + 16 \cdot 9 = 324. \end{aligned}$$

OR

$$\text{Var}(5X_2 + 4X_3) = (0 \ 5 \ 4) \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} = (-5 \ 28 \ 46) \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} = 324.$$

$$P(5X_2 + 4X_3 > 70) = P\left(Z > \frac{70-43}{\sqrt{324}}\right) = P(Z > 1.50) = \mathbf{0.0668}.$$

c) Find $P(3X_3 - 5X_2 < 17)$.

$$E(3X_3 - 5X_2) = 3 \cdot 7 - 5 \cdot 3 = 6.$$

$$\begin{aligned} \text{Var}(3X_3 - 5X_2) &= 9 \text{Var}(X_3) - 30 \text{Cov}(X_2, X_3) + 25 \text{Var}(X_2) \\ &= 9 \cdot 9 - 30 \cdot 2 + 25 \cdot 4 = 121. \end{aligned}$$

OR

$$\text{Var}(3X_3 - 5X_2) = (0 \quad -5 \quad 3) \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ 3 \end{pmatrix} = (5 \quad -14 \quad 17) \begin{pmatrix} 0 \\ -5 \\ 3 \end{pmatrix} = 121.$$

$$P(3X_3 - 5X_2 < 17) = P\left(Z < \frac{17-6}{\sqrt{121}}\right) = P(Z < 1.00) = \mathbf{0.8413}.$$

d) Find $P(4X_1 - 3X_2 + 5X_3 < 80)$.

$$E(4X_1 - 3X_2 + 5X_3) = 4\mu_1 - 3\mu_2 + 5\mu_3 = 4 \cdot 5 - 3 \cdot 3 + 5 \cdot 7 = 46.$$

$$\begin{aligned} \text{Var}(4X_1 - 3X_2 + 5X_3) &= (4 \quad -3 \quad 5) \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} = (19 \quad -6 \quad 39) \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} \\ &= 289. \end{aligned}$$

$$4X_1 - 3X_2 + 5X_3 \sim N(46, 289)$$

$$P(4X_1 - 3X_2 + 5X_3 < 80) = P\left(Z < \frac{80-46}{\sqrt{289}}\right) = P(Z < 2.00) = \mathbf{0.9772}.$$

e)* Find $P(X_1 > 8 \mid X_2 = 1, X_3 = 10)$.

$$\Sigma_{22} = \begin{pmatrix} 4 & 2 \\ 2 & 9 \end{pmatrix} \quad \Sigma_{22}^{-1} = \frac{1}{32} \begin{pmatrix} 9 & -2 \\ -2 & 4 \end{pmatrix}$$

$$\Sigma_{12} \Sigma_{22}^{-1} = \frac{1}{32} (-1 \quad 0) \begin{pmatrix} 9 & -2 \\ -2 & 4 \end{pmatrix} = \frac{1}{32} (-9 \quad 2)$$

$$\mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (\mathbf{X}_2 - \mu_2) = 5 + \frac{1}{32} (-9 \quad 2) \begin{pmatrix} 1-3 \\ 10-7 \end{pmatrix} = 5.75.$$

$$\Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} = 4 - \frac{1}{32} (-9 \quad 2) \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 3.71875.$$

$$X_1 \mid X_2 = 1, X_3 = 10 \sim N(5.75, 3.71875)$$

$$P(X_1 > 8 \mid X_2 = 1, X_3 = 10) = P\left(Z > \frac{8-5.75}{\sqrt{3.71875}}\right) = P(Z > 1.17) = \mathbf{0.1210}.$$